

Using Variational and PDE Methods for the Existence of Origami Models with Given Boundary Conditions

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Abstract

The purpose of this work is to develop a PDE-based tool for investigating the existence of an Origami model with given boundary conditions. An alternative approach for this goal is investigated by the authors in [3]. For the PDE approach, let a flat paper be parameterized by points $\mathbf{x} = (x, y) \in \Omega = [0, 1]^2 \subset \mathbb{R}^2$. Let the transformed paper be given by the image of a mapping $\mathbf{u} = (u, v, w) : \Omega \rightarrow \mathbb{R}^3$. Following [1], such a mapping may be modelled to be *locally rigid* in the sense that

$$\nabla \mathbf{u}(\mathbf{x})^\top \nabla \mathbf{u}(\mathbf{x}) = I, \quad (\text{for a.e.}) \quad \mathbf{x} \in \Omega,$$

i.e., except where \mathbf{u} is not regular such as at a fold. Since such a mapping is a local isometry [1], it preserves lengths, angles, areas and Gaussian curvature, and hence the surface $\mathbf{u}(\Omega)$ is developable. The aim is to investigate the solvability of this non-linear, first-order system of PDEs constrained by pre-specified boundary conditions on the paper edges,

$$\mathbf{u}(\mathbf{x}) = \mathbf{e}(\mathbf{x}), \quad (\text{for a.e.}) \quad \mathbf{x} \in \partial\Omega.$$

The approach is to minimize the following functional for ever decreasing regularization $\varepsilon > 0$,

$$J(\mathbf{u}) = \int_{\Omega} \left[\|\nabla \mathbf{u}(\mathbf{x})^\top \nabla \mathbf{u}(\mathbf{x}) - I\|_F^2 + 2\varepsilon \|\nabla^2 \mathbf{u}(\mathbf{x})\|_F^2 \right] d\mathbf{x}$$

under the constraint that \mathbf{u} satisfy the boundary conditions. Here, $\|\cdot\|_F$ denotes the Frobenius matrix norm and $\nabla^2 \mathbf{u}$ denotes the Hessian matrix of \mathbf{u} . Also, Δ^2 denotes the biharmonic operator. The optimality system for a minimizing \mathbf{u} is given by the non-linear, fourth-order system of PDEs,

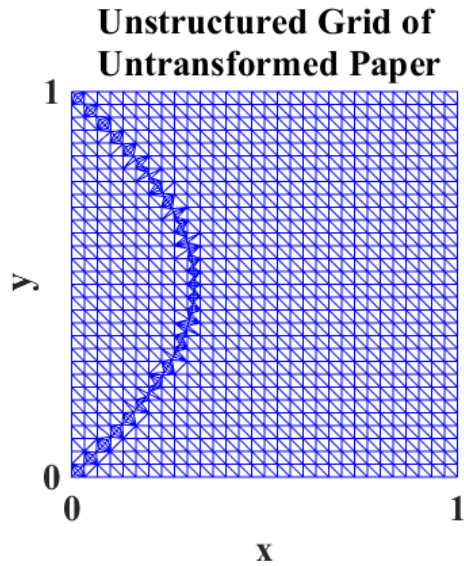
$$\begin{cases} [(\|\mathbf{u}_x\|^2 - 1)\mathbf{u}_x]_x + [(\|\mathbf{u}_y\|^2 - 1)\mathbf{u}_y]_y + [((\mathbf{u}_y^\top \mathbf{u}_x)\mathbf{u}_y)_x + ((\mathbf{u}_x^\top \mathbf{u}_y)\mathbf{u}_x)_y] - \varepsilon \Delta^2 \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{e} \quad \text{and} \quad \partial_n \nabla \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases}$$

which is solved using a Newton iteration,

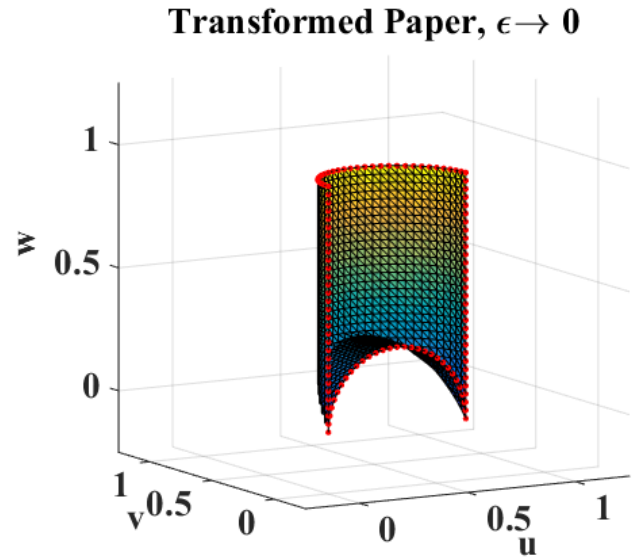
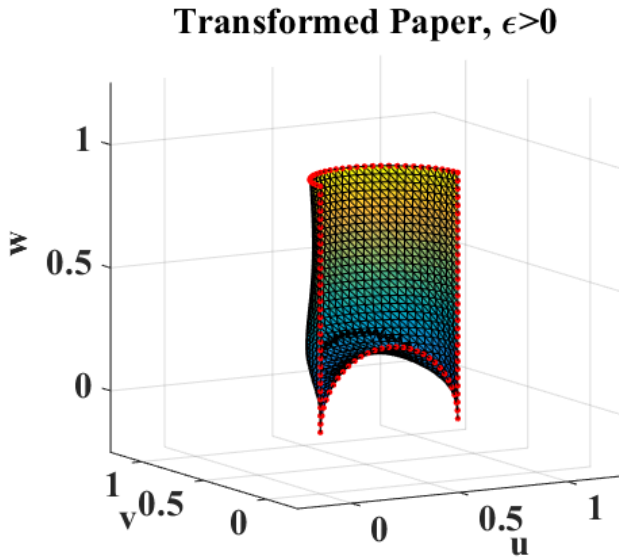
$$\partial_{\mathbf{u}}^2 J(\mathbf{u}^l; \mathbf{u}^{l+1} - \mathbf{u}^l, \bar{\mathbf{u}}) = -\partial_{\mathbf{u}} J(\mathbf{u}^l; \bar{\mathbf{u}}), \quad \forall \bar{\mathbf{u}} \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^3), \quad l = 0, 1, 2, \dots$$

Various approaches have been considered for the discretization of these PDEs. Finite differences are adequate, provided a fold conforms to a grid line. A rather more complicated but advantageous alternative is to employ finite elements on unstructured grids, which adapt to the irregularity of the solution as $\varepsilon \rightarrow 0$. This method is demonstrated below.

To test these methods, an example was used from [2], in which the paper is folded along a curve, and an exact solution is known. For the PDE approach, the unfolded paper was discretized with an unstructured grid of triangles shown in the following figure. Here triangles are concentrated



near the fold, and triangle vertices lie on the fold itself. In the next two figures, numerical solutions to the optimality system are shown first for a fixed, non-trivial value of ε (left) and then for ε vanishingly small (right). In the result on the left, the departure from local rigidity is evident from the global smoothness of the numerical solution. On the other hand, the result on the right manifests local rigidity, and its departure from the known solution is negligibly small.



As long as the regularization parameter ε is positive, a solution to the optimality system can be established. Furthermore, if there exists a locally rigid transformation satisfying the boundary conditions, then convergence of the solution to the optimality system as $\varepsilon \rightarrow 0$ can be established. If there is no locally rigid transformation satisfying the boundary conditions, then the solution to the optimality system diverges as $\varepsilon \rightarrow 0$. Thus, this tool may be used to investigate the existence of an Origami model with given boundary conditions.

References:

- [1] B. Dacorogna, P. Marcellini and E. Paolini, *Origami and Partial Differential Equations*, Notices of the AMS, Vol. 57, No. 5, pp. 598 – 606, Max, 2010.
- [2] R. Geretschl ger, *Folding Curves*, Proceedings of OSME4, 2009.
- [3] R. Geretschl ger and S. L. Keeling, *Using Direct and Constructive Methods for the Existence of Origami Models with Given Boundary Conditions*, Proceedings of OSME7, 2018.